Preparation for the Final.

Below you can find a list of definitions, axioms (well, an axiom), theorems and (counter-)examples that you need to know for the Final. More precisely, for in-class part of the Final, you will be given several (tentatively, ten to twenty) items from this list to formulate. Note that it won't have to be word-by-word citation, but whatever you write will need to be (a) correct (as in not a false statement), (b) easily equivalent to the textbook/lectures version.

On in-class part of the Final, you will *not* be asked to provide any proofs.

In the list below,
⊙ marks definitions and an axiom;
□ marks theorems and statements;
▷ marks (counter-)examples that you need to know off-hand.

Preliminaries.

⊙ Sets, set-theoretic operations, functions, inverse function, composition.

⊙ Injections, surjections, bijections.

⊙ Finite, infinite sets. Denumerable (countably infinite), countable, uncountable sets.

□ Countability of $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^n$, $\mathbb{Q}$.

□ Cantor's Theorem. Uncountability of $\mathbb{R}$.

Properties of $\mathbb{R}$.

⊙ Bounded, bounded above, bounded below subsets of $\mathbb{R}$.

⊙ Upper bound, lower bound of a subset of $\mathbb{R}$.

⊙ Least upper bound (= exact upper bound = supremum) of a subset of $\mathbb{R}$.

⊙ Greatest lower bound (= exact lower bound = infimum) of a subset of $\mathbb{R}$.

⊙ Completeness property of $\mathbb{R}$ (= supremum property of $\mathbb{R}$).

□ Archimedean property of $\mathbb{R}$.

□ Nested intervals property.

□ The density theorem.

Limits of Sequences.

⊙ Sequence of real numbers (= sequence in $\mathbb{R}$).

⊙ Limit of a sequence in $\mathbb{R}$, convergent/divergent sequence.

□ Uniqueness of limit of a sequence.

⊙ Tail of a sequence.

⊙ Bounded sequence.

□ Boundedness of a convergent sequence.

▷ Bounded but divergent sequence.

- ☐ Arithmetic properties of limits of sequences (Theorem 3.2.3).
- ▷ Divergent sequences A, B such that $A + B$ converges.
- ☐ Order properties of limits of sequences (Theorems 3.2.4, 3.2.5).
- ▷ Sequence (a_n) with $a_n > 0$ for all $n \in \mathbb{N}$, but $\lim(a_n) = 0$.
- ☐ Squeeze theorem for sequences.
- ☐ Increasing, strictly increasing, decreasing, strictly decreasing, monotone sequence.
- ☐ Monotone convergence theorem.
- ⊙ Euler's number e.
- ⊙ Subsequence of a sequence.
- ☐ Bolzano–Weierstrass theorem.
- ⊙ Cauchy sequence.
- ☐ Cauchy criterion.
- ⊙ Sequence that tends to $+\infty$, sequence that tends to $-\infty$, properly divergent sequence.
- ⊙ Sum of a series. Correspondence between infinite series and sequences.
- ☐ n-th term test.

Limits of Functions.
- ⊙ Cluster point of a subset of $\mathbb{R}$.
- ⊙ Limit of a function.
- ☐ Uniqueness of limit of a function.
- ☐ Sequential criterion for limit of a function.
- ⊙ Function bounded a neighborhood.
- ☐ Boundedness of a function that has a limit.
- ▷ Bounded function that does not have a limit at 0.
- ☐ Arithmetic properties of limits of functions (Theorem 4.2.4).
- ▷ Functions f, g that don't have a limit at some point $c \in \mathbb{R}$, but $f + g$ does.
- ☐ Order properties of limits of functions (Theorem 4.2.6).
- ▷ Functions f, g such that for all x in their domain, $f > g$, but at some point c, $\lim_{x \to c} f = \lim_{x \to c} g$.
- ☐ Squeeze theorem for limits of functions.
- ☐ Local separation from zero (Theorem 4.2.9).
- ⊙ Infinite limit of a function, limit of a function at infinity, infinite limit of a function at infinity.
- ⊙ One-sided limits.

Continuous Functions.
- ⊙ Function, continuous at a point. Function, discontinuous at a point.

☐ Criterion for continuity in terms of neighborhoods (Theorem 5.1.2).

☐ Sequential criterion for continuity.

☐ Sequential criterion for discontinuity.

⊙ Function, continuous on a subset of $\mathbb{R}$.

▷ Function, discontinuous everywhere (for example, Dirichlet's function).

▷ Function, continuous at irrational numbers and discontinuous at rational numbers (for example, Thomae's function).

☐ Arithmetic properties of continuous functions (Theorem 5.2.1).

▷ Functions f, g discontinuous at 0 such that $f + g$ is continuous at 0.

☐ Composition of continuous functions (at a point and on a set).

☐ Boundedness Theorem.

▷ Bounded but discontinuous (at least at one point) function.

▷ Function continuous but unbounded on an open interval.

⊙ Absolute (= global) maximum of a function on a set, point of absolute maximum. Absolute minimum of a function on a set, point of absolute minimum.

☐ Maximum–Minimum Theorem.

▷ Function f continuous on an open interval such that that f does not have maximum or minimum value.

☐ Location of roots theorem, Bolzano's intermediate value theorem.

☐ Preservation of closed intervals. Preservation of intervals.

⊙ Function, uniformly continuous on a subset of $\mathbb{R}$.

☐ Nonuniform continuity criterion.

☐ Uniform continuity theorem.

▷ Function, continuous but not uniformly continuous on an open interval.

⊙ Increasing, strictly increasing, decreasing, strictly decreasing, monotone functions.

⊙ Jump of a monotone function.

☐ Continuity criterion of monotone functions (Theorem 5.6.3).

☐ Continuous inverse theorem.

Differentiation.

⊙ Derivative of a function at a point. Function, differentiable at a point.

☐ Continuity of a differentiable function.

▷ Function, continuous but not differentiable at $x = 0$.

▷ Function, differentiable on $\mathbb{R}$, whose derivative is not continuous at 0.

☐ Arithmetic properties of derivative.

☐ Chain rule.

- ☐ Derivative of inverse function (inverse function theorem).
- ☐ Interior extremum theorem.
- ☐ Rolle's theorem.
- ☐ Mean value theorem.
- ☐ First derivative test for extrema (Theorem 6.2.8).
- ☐ Criterion for a differentiable function to be increasing/decreasing/constant on an interval (Theorems 6.2.5, 6.2.7).
- ⊙ nth Taylor polynomial of a function.
- ☐ Taylor's theorem.
- ☐ nth derivative test for extrema (Theorem 6.4.4).
- ☐ nth Taylor's polynomial at zero for $(1+x)^{\alpha}$, e^x, $\ln x$, $\sin x$, $\cos x$.

The Riemann Integral.

- ⊙ Partition, tagged partition, Riemann sum.
- ⊙ Riemann integrable function, Riemann integral.
- ▷ Not a Riemann integrable function.
- ☐ Arithmetic and order properties of Riemann integral.
- ☐ Boundedness theorem for Riemann integrable function.
- ☐ Cauchy Criterion for Riemann integral.
- ☐ Riemann integrability of a step function, of a continuous function, monotone function.
- ☐ Interval additivity theorem.
- ☐ The fundamental theorem (first form).
- ⊙ Indefinite integral.
- ☐ The fundamental theorem (second form).
- ▷ Integrable (but discontinuous) function f such that $(\int_a^x f)' \neq f$ at least at one point.
- ☐ Derivative of an indefinite integral of a continuous function.
- ☐ Substitution theorem.
- ⊙ Null (or Lebesgue measure zero) set.
- ☐ Lebesgue's integrability criterion.
- ☐ Composition theorem.
- ☐ The product theorem.
- ☐ Integration by parts.
- ☐ Existence of exponent. Power function.
- ☐ Existence of sine and cosine.